Eighth Annual Upper Peninsula
High School Math Challenge

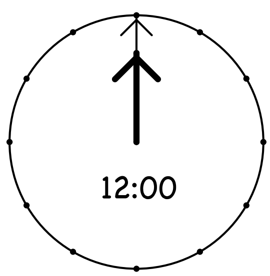
Northern Michigan University (Marquette, MI, USA)
Saturday 8 April 2017

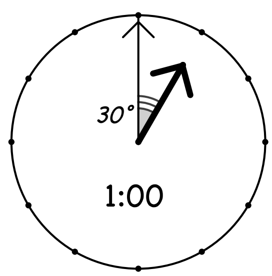
Team Problems — Solutions

1. A regular analog clock with continuously moving hour and minute hands starts moving when both hands are together at noon. How many minutes does it take for the minute hand to next catch the hour hand?

Answer: $65\frac{5}{11}$ minutes

(Minute hand moves 6° per minute; hour hand moves 0.5° per minute.)





Time	M@	H@
1:00	0°	30°
1:01	6°	30.5°
1:02	12°	31°
1:03	18°	31.5°
1:04	24°	32°
1:05	30°	32.5°
1:06	36°	33°

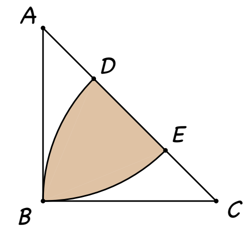
time to alignment = t seconds
angular distance = d degrees
hour hand: $d = 0.5 \cdot t$
minute hand: $360 + d = 6 \cdot t$

$360 + 0.5t = 6t$
$360 = 5.5t$
$t = \dfrac{360}{5.5} = 65.454545\ldots$minutes

$\left(\text{i.e., } 65\frac{5}{11} \text{ minutes} \Rightarrow 1{:}05{:}\frac{5}{11}\right)$

2. Triangle ABC is an isosceles right triangle with BC = AB = 2 inches. Circular arcs of radius 2 inches centered at C and A meet the hypotenuse at D and E, respectively. What is the area of the shaded region?

(Express your answer in terms of π and/or radicals, if appropriate. Do not approximate as a decimal.)



Answer: $(\pi - 2)\ \text{in}^2$

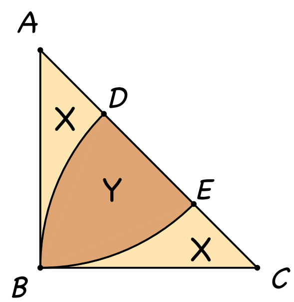
$$\text{area } \triangle ABC = \frac{1}{2}(2 \cdot 2) = 2\ \text{in}^2$$

$\triangle ABC$ isoscelese $\Rightarrow \angle A = \angle C = 45°$

$\therefore \overset{\frown}{BE} = \overset{\frown}{DB} = 45°$

$$\text{area sector } \overset{\frown}{ABE} = \text{area sector } \overset{\frown}{CDB} = \frac{\pi \cdot 2^2}{8} = \frac{\pi}{2}$$

$$\text{area sector } \overset{\frown}{ABE} = X + Y = \frac{\pi}{2}$$



$$\text{area sector } \overset{\frown}{CDB} = X + Y = \frac{\pi}{2}$$

$$\text{area } \triangle ABC = 2X + Y = X + (X + Y)$$

$$= X + \frac{\pi}{2} = 2$$

$$X = 2 - \frac{\pi}{2}$$

$$\text{area } \overset{\frown}{ABE} + \text{area } \overset{\frown}{CDB} = 2X + 2Y = \text{area } \triangle ABC + Y$$

$$\frac{\pi}{2} + \frac{\pi}{2} = 2 + Y$$

$$\pi - 2 = Y$$

3. For a given arithmetic sequence, the sum of the first fifty terms is 200, and the sum of the next fifty terms is 2700. What is the first term of the sequence?

<u>Answer: −20.5</u>

Let ***a*** be the first term of the sequence, and let ***d*** be the common difference between consecutive terms. Then the sum of the first 50 terms is

$$a + (a+d) + (a+2d) + \cdots + (a+49d) =$$
$$= 50a + d\left(1 + 2 + \ldots + 49\right)$$
$$= 50a + \frac{d(49)(50)}{2}$$
$$= 50a + 1225d \qquad (1)$$

and the sum of the first 100 terms is $100a + \frac{d(99)(100)}{2} = 100a + 4950d$ (2)

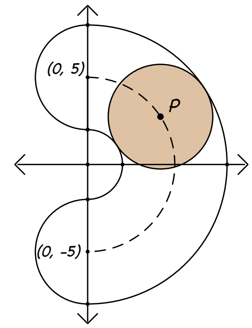
$\therefore$ sum of the second 50 terms is $(100a + 4950d) - (50a + 1225d) = 50a + 3725d$. (3)

$\therefore$ $50a + 1225d = 200$ and $50a + 3725d = 2700$
$\Rightarrow$ $(50a + 3725d) - (50a + 1225d) = 2500d = 2700 - 200 = 2500$ (3)−(1) = (4)

$\therefore$ $2500d = 2500 \Rightarrow d = 1$

$\therefore$ $50a + 1225 = 200 \Rightarrow 50a = -1025$, so the first term of the sequence is $\underline{a = -20.5}$

4. The center of circle P travels along a semi-circular path from (0, 5) to (0, −5). If the circle sweeps out an area of 39π units^2, what is the radius of circle P?



<u>Answer: 3 units</u>

The area swept out by circle P consists of half an annulus (ring) plus two semicircles. Let r be the radius of the circle. Then the outer radius of the annulus is 5 + r, and its inner radius is 5 − r.

The area of the half-annulus is:

$$\frac{\pi}{2}\left[(5+r)^2-(5-r)^2\right]=\frac{\pi}{2}\left[25+10r+r^2-\left(25-10r+r^2\right)\right]$$

$$=\frac{\pi}{2}(20r)=10\cdot\pi\cdot r$$

Add the two semicircles to obtain $\pi r^2 + 10\pi r$.

Then $\pi r^2 + 10\pi r = 39\pi$

$$r^2 + 10r - 39 = 0$$

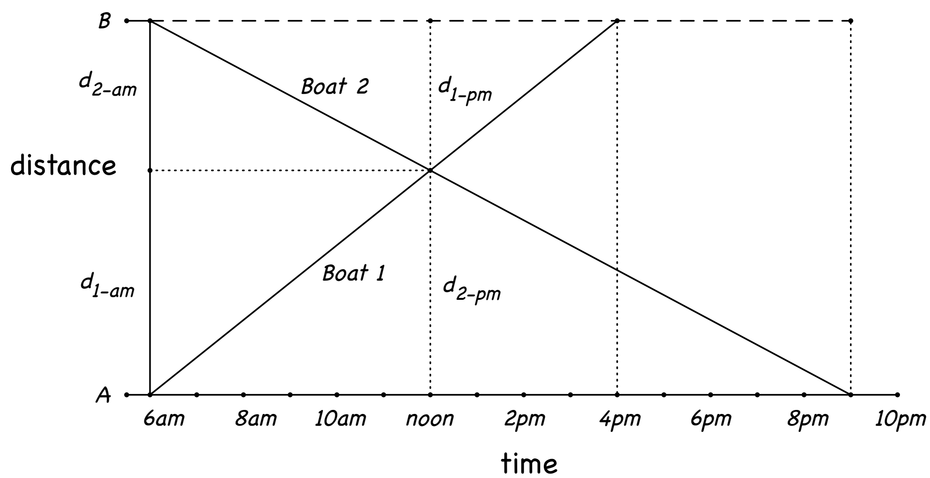
$$r=\frac{-10\pm\sqrt{100-(4)(-39)}}{2}$$

$$=\frac{-10\pm\sqrt{256}}{2}=\frac{-10\pm16}{2}$$

r = −13 or 3. We reject the negative solution.

5. Two boats traveled in opposite directions (north and south) along the Mississippi River, each at a constant speed. Both boats left at exactly the same time, at sunrise. One boat went from A to B and arrived at 4 pm. The other boat went from B to A and arrived at 9 pm. If they passed each other at noon, what time was sunrise that day?

<u>**Answer: 6 am**</u>



Before noon:
distance traveled by B_1 before noon = distance traveled by B_2 after noon
distance traveled by B_2 before noon = distance traveled by B_1 after noon

$d_{1\text{-}am} = d_{2\text{-}pm} \qquad d_{1\text{-}am} = r_1 \cdot t = 9r_2 \qquad t = \dfrac{9r_2}{r_1}$

$d_{2\text{-}am} = d_{1\text{-}pm} \qquad d_{2\text{-}am} = r_2 \cdot t = 4r_1 \qquad t = \dfrac{4r_1}{r_2}$

$$\frac{9r_2}{r_1} = \frac{4r_1}{r_2} \Rightarrow 9(r_2)^2 = 4(r_1)^2 \Rightarrow 3r_2 = 2r_1$$

$r_2 = \dfrac{2}{3}r_1 \qquad\qquad r_1 = \dfrac{3}{2}r_2$

$t \cdot \left(\dfrac{2}{3}r_1\right) = 4r_1 \qquad\qquad t \cdot \left(\dfrac{3}{2}r_2\right) = 9r_2$

$t = \dfrac{12}{2} = 6 \qquad\qquad t = \dfrac{18}{3} = 6$

$t = 6$ hours $\Rightarrow$ sunrise at 6 a.m.